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(1a) $\underset{z=0}{\text{Res}} \dfrac{1}{z+z^2} = \underset{z=0}{\text{Res}} \dfrac{1}{z(1+z)} = \lim_{z\to 0} \dfrac{1}{z(1+z)} \cdot z = 1$

(1b) $z\cos\dfrac{1}{z} = z\sum_{n=0}^{\infty} \dfrac{(-1)^n z^{-2n}}{(2n)!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{-2n+1}}{(2n)!}$

$\therefore \underset{z=0}{\text{Res}}\, z\cos\dfrac{1}{z} = \dfrac{-1}{2}$

(1c) $\underset{z=0}{\text{Res}} \dfrac{z-\sin z}{z} = \lim_{z\to 0} \dfrac{z-\sin z}{z} \cdot z = 0$

(1d) $\cos z = 1 - \dfrac{z^2}{2} + \dfrac{z^4}{24} + \cdots, \quad \dfrac{1}{\sin z} = \dfrac{1}{z} + \dfrac{z}{6} + \dfrac{7z^3}{360} + \cdots$

$\cot z = \dfrac{1}{z} - \dfrac{z}{3} - \dfrac{z^3}{45} + \cdots$

Hence, $\underset{z=0}{\text{Res}} \dfrac{\cot z}{z^4} = \dfrac{-1}{45}$

(1e) $\dfrac{\sinh z}{1-z^2} = \left(z + \dfrac{z^3}{6} + \dfrac{z^5}{120} + \cdots\right)\left(1 + z^2 + z^4 + \cdots\right)$

$= z + \dfrac{7z^3}{6} + \cdots$

Hence, $\underset{z=0}{\text{Res}} \dfrac{\sinh z}{z^4(1-z^2)} = \dfrac{7}{6}$

(2a) $e^{-z} = 1 - z + \dots$ $\quad \operatorname*{Res}_{z=0} \frac{e^{-z}}{z^2} = -1$

$$\int_C \frac{e^{-z}}{z^2} = 2\pi i(-1) = -2\pi i$$

(2b) $e^{-z} = e^{-1}e^{-(z-1)} = e^{-1}\left(1-(z-1)+(z-1)^2/2+\dots\right)$

$$\operatorname*{Res}_{z=1} \frac{e^{-z}}{(z-1)^2} = -e^{-1}$$

$$\int_C \frac{e^{-z}}{(z-1)^2} = 2\pi i(-e^{-1}) = \frac{-2\pi i}{e}$$

(2c) $e^{1/z} = \left(1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\dots\right)$, $\operatorname*{Res}_{z=0} z^2e^{\frac{1}{z}} = \frac{1}{6}$

$$\int_C z^2e^{\frac{1}{z}} = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

(2d) $\operatorname*{Res}_{z=0} \frac{z+1}{z^2-2z} = \lim_{z\to 0} \frac{z+1}{z^2-2z}\cdot z = -\frac{1}{2}$

$\operatorname*{Res}_{z=2} \frac{z+1}{z^2-2z} = \lim_{z\to 2} \frac{z+1}{z^2-2z}(z-2) = \frac{3}{2}$

$$\int_C \frac{z+1}{z^2-2z} = 2\pi i\left(\frac{3}{2}-\frac{1}{2}\right) = 2\pi i$$

(3) $\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{\frac{4}{z}-5}{\frac{1}{z}\left(\frac{1}{z}-1\right)}\right) = \frac{4-5z}{z-z^2}$

$\operatorname*{Res}_{z=0} \frac{1}{z^2}f\left(\frac{1}{z}\right) = \lim_{z\to 0} \frac{4-5z}{z-z^2}\cdot z = 4$

$$\int_C \frac{4z-5}{z(z-1)}dz = 8\pi i$$

(4a) $\operatorname*{Res}_{z=0} \frac{z^{-5}}{1-z^{-3}}\cdot\frac{1}{z^2} = \operatorname*{Res}_{z=0} \frac{1}{z^4}\left(\frac{1}{z^3-1}\right)$

$\frac{1}{1-z^3} = 1+z^3+z^6+\dots$

the required residue is -1,

Hence $\int_C \frac{z^5}{1-z^3} = -2\pi i$

(4b) $\underset{z=0}{\text{Res}} \frac{1}{z^2}\left(\frac{1}{1+z^{-2}}\right) = \underset{z=0}{\text{Res}}\left(\frac{1}{z^2+1}\right) = 0$.

$\int_C \frac{1}{1+z^2} = 0$

(4c) $\underset{z=0}{\text{Res}} \frac{1}{z^2}(z) = \underset{z=0}{\text{Res}} \frac{1}{z} = 1$

$\int_C \frac{1}{z} = 2\pi i$

(5a) $\int_C e^{z+\frac{1}{z}} = \int_C \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}\int_C z^n e^{\frac{1}{z}}$

(5b) $\underset{z=0}{\text{Res}}\ z^n e^{1/z} = \frac{1}{(n+1)!}$, Result follows.

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(1a) Principal Part: $\sum_{n=1}^{\infty} \frac{1}{z^n (n+1)!}$

$z=0$ is essential singular point.

~~(1b) Principal Part:~~

(1b) $\frac{z^2}{1+z} = \frac{(z+1-1)^2}{z+1} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1}$

Principal part: $\frac{1}{z+1}$

$z=-1$ is pole of order 1.

1c Principal part: 0 , it is removable singularity

1d Principal part: $1/z$, it is pole of order 1.

1e Principal part: $-1/(z-2)^3$, it is pole of order 3.

2a $$\frac{1-\cosh z}{z^3} = \sum_{n=1}^{\infty} \frac{-z^{2n}}{(2n)!\,z^3} = \sum_{n=1}^{\infty} \frac{-z^{2n-3}}{(2n)!}$$

$m=1, \quad B=-1/2$

2b $$\frac{1-e^{2z}}{z^4} = \sum_{n=1}^{\infty} \frac{-(2z)^n}{n!\,z^4} = \sum_{n=1}^{\infty} \frac{-2^n}{n!} z^{n-4}$$

$m=3, \quad B=-4/3$

2c $$\frac{e^{2z}}{(z-1)^2} = \frac{e^{2(z-1)}e^2}{(z-1)^2} = e^2 \sum_{n=0}^{\infty} \frac{(2(z-1))^n}{n!\,(z-1)^2} = e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!}(z-1)^{n-2}$$

$m=2, \quad B=2e^2.$

3a Suppose $f(z)$ can be represented by the Taylor series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ with } a_n = \frac{f^{(n)}(z_0)}{n!}$$

If $f(z_0) = a_0 \neq 0$, then

$$f(z) = \frac{f(z_0)}{z-z_0} + \sum_{n=0}^{\infty} a_{n+1}(z-z_0)^n$$

Hence z_0 is simple pole.

(3b) $\lim_{z\to z_0} g(z)(z-z_0) = \lim_{z\to z_0} f(z) = f(z_0) = 0.$

Hence it is removable.

(4) ϕ has Taylor series about $z=ai$ since $8a^3z^2$ and $\frac{1}{(z+ai)^3}$ are analytic about $z=ai$.

$$f(z) = \left(\frac{1}{z-ai}\right)^3 \sum_{n=0}^{\infty} \frac{\phi^{(n)}(ai)}{n!}(z-ai)^n$$

So its principal part is $\frac{\phi''(ai)}{2(z-ai)} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi(ai)}{(z-ai)^3}$

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(5a) $\operatorname{Res}_{z=\pm\frac{\pi}{2}} \tan z = \lim_{z=\pm\frac{\pi}{2}} \frac{\sin z}{\cos z}\cdot\left(z\mp\frac{\pi}{2}\right) = (\pm 1)\left(\lim_{z\to\pm\frac{\pi}{2}} \frac{(z\mp\pi/2)}{\cos z}\right)$

$= -1$

$$\int_C \tan z\, dz = -4\pi i$$

(5b) $\sinh 2z = 0$ iff $z = \frac{i\pi n}{2}$, $n\in\mathbb{Z}$

$$\operatorname{Res}_{z=\frac{i\pi n}{2}} \frac{1}{\sinh 2z} = \lim_{z\to\frac{i\pi n}{2}} \frac{z-i\pi n/2}{\sinh 2z} = \lim \frac{1}{2\cosh 2z} = \frac{1}{2\cos\pi n} = \frac{1}{2(-1)^{|n|}}$$

$$\therefore \int_C \frac{1}{\sinh 2z} = 2\pi i\left(\operatorname{Res}_{z=0} + \operatorname{Res}_{z=\frac{\pi}{2}i} + \operatorname{Res}_{z=-\frac{\pi i}{2}} \frac{1}{\sinh 2z}\right) = -\pi i$$

(6)

Since $\int_{C_N} \frac{1}{z^2 \sin z} = \operatorname*{Res}_{z=0} \frac{1}{z^2\sin z} + \sum_{n=1}^{N} \operatorname*{Res}_{z=\pm n\pi}\left(\frac{1}{z^2 \sin z}\right)$

About $z=0$, $\frac{1}{z^2\sin z} = \frac{1}{z^2}\left(\frac{1}{z} + \frac{z}{6} + \dots\right)$

$$\operatorname*{Res}_{z=0} \frac{1}{z^2 \sin z} = \frac{1}{6}$$

$$\operatorname*{Res}_{z=\pm n\pi}\left(\frac{1}{z^2\sin z}\right) = \lim_{z\to \pm n\pi} \frac{z-(\pm n\pi)}{z^2 \sin z}$$

$$= \left(\frac{1}{n\pi}\right)^2 \frac{1}{\cos(n\pi)}$$

$$= \frac{(-1)^n}{n^2\pi^2}$$

Thus $\int_{C_N} \frac{1}{z^2\sin z} = 2\pi i\left(\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right)$

Since $\int_{C_N} \frac{1}{z^2 \sin z} \to 0$ as $N \to \infty$, Result follows.